Demography and Growth: A Unified Treatment of Overlapping Generations*

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Abstract

We construct a unified overlapping-generations (OLG) framework of equilibrium growth that includes the Blanchard "perpetual youth" model, the Samuelson model, and the infinitely-lived representative agent growth model as limit specifications of a "realistic", two-parameter survivorship function. We analyze how exogenous changes in demographic conditions affect the equilibrium growth and savings rates by computing equilibrium rates under different specifications of the survivorship function. Differences in population growth rates, life-expectancies, retirement durations, and the degree of concavity of the survivorship function are found to have significant impacts on equilibrium growth rates. The observed effects are consistent with some cross-country correlations between demographic conditions and growth rates. We also identify a potential "Malthusian growth trap" in economies where life expectancy is short, fertility rates are high, and households work most of their lives—conditions often found in less developed economies.

Keywords: Demography, Growth, Overlapping Generations

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1. Introduction

Most economists would agree that the demography of an economy is an important determinant of its growth potential and performance. This concern is often expressed in the context of “ageing economies” and the financing of the impending retirement of the “baby boom” generation. Yet despite the acknowledged importance of demographic issues, contemporary economic growth theory has not addressed them in a unified way. The standard benchmark growth models remain based on infinitely-lived representative agent models, which lack the structure necessary to address demographic issues.[1]

Two workhorse models that do incorporate demographic features (overlapping generations) are the Samuelson (1958) and Blanchard (1985) models, both of which provide deep insights and have had profound impact.[2] However, both are highly stylized, which limits their ability to incorporate demographic factors in a comprehensive way. The basic Samuelson model usually adopts a two-period framework—period one for working and period two for retirement—although extensions to an initial third period, for education, also exist.[3] While the discrete-time Samuelson model can be used to analyze many inter-generational policy issues, the usual formulation is overly inflexible with regard to its choice of time units. For example, as typically specified, the Samuelson model (implicitly) assumes that an agent’s working and retirement periods are fixed and of equal length (one time unit each). It therefore does not allow for the changing length of the retirement period relative to the working period, an important policy issue in many economies, and a factor that turns out to be an important determinant of the long-run growth rate. Also, identifying time units with generations renders the model cumbersome for policy analysis. As a result, Auerbach and Kotlikoff (1987), in their comprehensive study of fiscal policy, introduced 55 periods in order to accommodate several generations while employing a plausible time unit.

[1] We have in mind the Ramsey model or some form of the Romer (1986) model, depending upon the underlying production structure.
[2] The Samuelson model is often coupled with Diamond (1965), while Blanchard is sometimes linked with Yaari (1965) and Weil (1985). A comprehensive treatment of the Samuelson and Blanchard models and their applications to issues in economic growth is provided de la Croix and Michel (2002).
[3] See e.g. Docquier and Michel (1999).

The Blanchard model is simpler and more amenable to a growth framework, but this comes at a price. This formulation assumes an exponential survivorship function that has a single parameter—a mortality hazard rate that is independent of the household's age. While this "perpetual youth" assumption is convenient for analytical tractability, it is at odds with the facts of human mortality, which exhibit senescence (a mortality hazard rate that increases with age).[4]

We develop a unified treatment of overlapping generations and economic growth that allows for more general, and plausible, mortality assumptions and therefore a richer demographic framework. To do this we utilize a two-parameter survivorship function due to de Moivre (1725). The de Moivre survivorship function is tractable, yet fits the main characteristics of modern human mortality quite well, except at the old-age tail of its distribution.[5] Moreover, it includes both the Samuelson and Blanchard OLG models as limiting cases. This enables us to nest the two classic OLG models, along with the conventional representative agent growth model, as particular parameter specifications of a more general demographic structure. Nesting the OLG models within a unified framework, rather than presenting them (as is typically done) as alternative approaches, enhances our understanding of how demographic conditions affect the economic growth rate.[6]

In order to develop a tractable model with general demographic assumptions, we maintain a simplified production side economy, and assume that output is produced according to an "AK" production function where the return to capital is constant and the equilibrium economy is always on its balanced growth path.[7] After characterizing the equilibrium for a general survivorship function,

[4] Because of its tractability there is a substantial literature introducing the Blanchard mortality structure to growth models; see e.g. Saint Paul (1992), Heijdra and Ligthart (2006), and most recently, Tamai (2009). Bommier and Lee (2003) derive a number of propositions for overlapping generations models with "realistic demography" in exchange economies and in economies without technical progress. Boucekkine *et al* (2002) also develop a human capital growth model that utilizes a mortality function that exhibits increasing mortality hazard with age.

[5] Demographers commonly use the two-parameter Gompertz (1825) mortality hazard function, which fits human mortality data well. However, the Gompertz survivorship function is intractable for analytical purposes. Neither the Gompertz nor the de Moivre survivorship functions exhibit high infant mortality rates. However, this phenomenon has largely been eliminated in advanced economies. Moreover, infant mortality can be easily modeled as a lower birth rate.

[6] For example, in motivating the perpetual youth model as an alternative approach, Blanchard and Fischer (1989, p.115) argue that "overlapping generations models with more than two generations are analytically intractable". More recently, in introducing the OLG model, Acemoglu (2009, p.327) characterizes the Blanchard perpetual youth model as "a tractable alternative to the basic OLG model." We should note also that there is a literature analyzing continuous-time overlapping generations economies, with finite horizons, that originated with Cass and Yaari (1967). This literature tends to focus on issues related to existence of equilibrium and its characterization in a more abstract context than we have in mind here; see e.g. Burke (1996), d'Albis (2007), Edmond (2008), Gan and Lau (2010).

[7] Deriving the transitional dynamics for growth models having more realistic demographic structures is extremely challenging and except for very special cases intractable; see d'Albis and Augeraud-Véron (2009). For this reason, we

we parameterize the model using the de Moivre function and solve explicitly for the equilibrium growth rates under different demographic conditions. Except for the Samuelson and Blanchard specifications, we are unable to express closed form analytical solutions for the equilibrium growth rate, but we can numerically solve for the equilibrium growth rates given parameter values that represent the different demographic conditions.

Using comparable calibrations, we compute equilibrium growth rates for the Blanchard and Samuelson specifications, as well as for intermediate specifications. Household mortality, alone, leads to lower equilibrium growth rates than those generated by the infinitely-lived, representative agent model. Indeed, without retirement, in the Samuelson specification growth is eliminated entirely for realistic life expectations. In contrast, when household mortality is combined with a realistic, but exogenous, retirement duration, economic growth rates exceed those predicted by the representative agent model.[8] Using our model, we allow for widely varying demographic conditions and compute the effects of these demographic changes on the equilibrium growth rate. These exogenous changes include varying the life expectancy, population growth and birth rates, and the retirement duration. An increase in the population growth rate, whether from higher fertility or reduced mortality, reduces the economic growth rate. However, economic growth is increased when longer life expectancy results in longer retirement. Holding the population growth rate constant, an increase in life-expectancy has non-monotonic effects on the economic growth rate, yielding an inverted U-shape relationship between growth and aging that has been found in some cross-country studies. We also show that economies with high fertility rates, short life expectancy, and no retirement have low or negative economic growth rates. This suggests that the demographic conditions prevailing in some less developed economies may create a "Malthusian trap" of low or negative growth rates.

A key role in our analysis is played by the shape of the survivorship function in terms of its concavity/convexity property. Increased concavity of this function, which corresponds to greater

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note that Heijdra and Mierau (2010) also employ the Romer technology in their analysis of consumption and labor income taxation, in which case their economy is also always on its balanced growth path. Furthermore, d'Albis (2007), who introduces a general demographic structure into a neoclassical growth model, is restricted to characterizing its steady state behavior.

[8] Note that the standard formulation of the discrete-time Samuelson model assumes that each generation is retired for half of its life span; see e.g. Blanchard and Fischer (1989).

certainty of reaching old age, has significant impacts on the growth rate. From this standpoint, the crucial difference between the Blanchard and Samuelson specifications is that the former specifies a convex survivorship function, while the latter adopts an extreme form of concavity.

The remainder of the paper is structured as follows. Section 2 lays out the basic analytical framework for consumption-saving decisions by households who are subject to aging, while Section 3 describes the corresponding demographic structure, derives descriptive expressions for the aggregate economy, and explains the solution methods. Section 4 parameterizes the demographic functions using the de Moivre functional form, and recapitulates the equations of the model. Section 5 performs numerical simulations. These involve computing the equilibrium growth and savings rates for various specifications of the de Moivre function, including the Samuelson and Blanchard models as limiting cases. Section 6 concludes, while the Appendix explicitly solves the model for the Samuelson and Blanchard models and shows that both converge to the representative agent model as lives (and working lives) become arbitrarily long.

2. The Analytical Framework

In an overlapping generations framework, it is necessary to distinguish clearly between household age and calendar time.[9] To avoid potential confusion between these two time concepts, we adopt a particular notational convention. Specifically, household variables are indexed in parentheses by age (indexes may be x, y, or z). Where a variable depends on calendar time, the variable is indexed by the means of subscripts. Thus, for example, $v_t(x)$ denotes the value of variable v at time t for a household of age x. When household indexes are absent, the time subscript denotes an economy-wide value of the variable at the subscripted time. The current time is denoted t, so, for example, w_t denotes the value of the variable w prevailing in the economy at the current time. The absence of an age index always indicates that the variable is independent of age. Aggregate variables, obtained by summing over cohorts, depend on calendar time only.

[9] In the infinitely-lived representative agent framework these concepts coincide, so that the distinction is irrelevant.

2.1 Households

In this section, we develop the consumption-saving behavior of a household with a general survivorship function. Let $S(z)$ denote the probability at birth of the household surviving to age *z* and ω the maximum attainable age. Because survivorship declines with age, $S'(z) < 0$, $0 < z < \omega$, with $S(z) = 0$, for $z \geq \omega$. With this notation, $S(z)/S(x)$ is the probability of surviving to age *z* conditional on surviving to age *x*, while $-S'(z)/S(z)$ is the mortality hazard rate at age *z*.

We focus initially on a household of age *x*, at time *t*, and assume that this unit maximizes its expected utility over the remainder of its life, that is:

$$U_t(x) = \int_{z=x}^{z=\omega} \frac{S(z)}{S(x)} \cdot e^{-\rho\cdot(z-x)} \cdot u\left(c_t(z)\right) \cdot dz \tag{1a}$$

where ρ is the pure time discount rate, and $c_t(z)$ is its planned consumption at age *z*. The household's flow budget constraint at age *z* is

$$\frac{df_t(z)}{dz} \equiv f_t'(z) = i_t(z) \cdot f_t(z) + w_t(z) \cdot L(z) - c_t(z) \tag{1b}$$

where $w_t(z)$ is the market wage facing the household at age *z*, $f_t(z)$ is the household's financial wealth, $i_t(z)$ is the interest rate on financial wealth facing the household at age *z*, and $L(z) \leq 1$ is the fraction of the household's unit time endowment supplied as labor at age *z*. Typically, households reduce the fraction of time spent working as they age, thus $L'(z) \leq 0$.[10] Under production conditions introduced later in the paper, the interest rates facing households are independent of time, so $i_t(z) = i(z)$ the budget constraint (1b) can be expressed equivalently

$$\frac{d}{dz}\left(R(z,x) \cdot f_t(z)\right) = R(z,x) \cdot \left[w_t(z) \cdot L(z) - c_t(z)\right] \tag{1b'}$$

where $R(z,x) \equiv \mathrm{e}^{-\int_{y=x}^{y=z} i(y)\cdot dy}$ is the discount factor for a flow at age *z* to a household at age *x*.

[10] We assume that $L(x)$ is specified exogenously. While the function can be quite general, allowing for either abrupt or gradual "retirement", we assume that it does not vary with (calendar) time.

Defining the present value Hamiltonian

$$H_t(z) \equiv e^{-\rho\cdot(z-x)}\frac{S(z)}{S(x)}\left\{u\left(c_t(z)\right)+\phi_t(z)\cdot\left[w_t(z)\cdot L(z)+i(z)\cdot f_t(z)-c_t(z)\right]\right\} \qquad (2)$$

and optimizing with respect to $c_t(z)$ and $f_t(z)$, we obtain the first order conditions

$$u'\left(c_t(z)\right)=\phi_t(z) \qquad (3a)$$

$$\rho-\frac{\phi_t'(z)}{\phi_t(z)}-\frac{S'(z)}{S(z)}=i(z). \qquad (3b)$$

Equation (3a) equates the marginal utility of consumption to the shadow value of financial wealth, while (3b) equates the rate of return on consumption, modified by the mortality hazard rate, to the rate of return on financial assets. In addition, the agent must satisfy the transversality condition, which for the agent having a maximum lifespan of ω is[11]

$$R(\omega,x).f_t(\omega)=0 \qquad (3c)$$

Following much of the growth literature, we assume an iso-elastic utility function of the form $u\left(c_t(z)\right)=c_t(z)^{\varepsilon}/\varepsilon$ ($\varepsilon\le 1$) where $1/(1-\varepsilon)$ is the inter-temporal elasticity of substitution, enabling us to rewrite (3b) as

$$\frac{c_t'(z)}{c_t(z)}=\frac{1}{(1-\varepsilon)}\cdot\left(i(z)-\rho+\frac{S'(z)}{S(z)}\right). \qquad (4)$$

We follow Blanchard (1985) and Yaari (1965) in assuming that when mortality hazard is present, households invest wholly in actuarially fair life annuities, so that

$$-\frac{\partial R(z,x)/\partial z}{R(z,x)}=i(z)=r-\frac{S'(z)}{S(z)} \qquad (5)$$

where *r* is the risk-free rate of return on capital, and $-S'(z)/S(z)$ is the mortality hazard premium for a household at age *z*. We assume that *r* is constant, an assumption that is duly validated under the

[11] As $\omega\to\infty$, the transversality condition converges to the conventional expression $\lim_{\omega\to\infty} R(\omega,x).f_t(\omega)=0$.

assumption of a Romer (1986) endogenous growth technology. Combining (4) and (5) yields

$$\frac{c_t'(z)}{c_t(z)} = \frac{1}{(1-\varepsilon)} \cdot (r-\rho) \tag{6}$$

Equation (6) expresses how household consumption changes with its age. Along with the marginal product of capital, this expression will be recognized as determining the equilibrium growth rate in the infinitely-lived representative agent growth model.[12] As we show and emphasize later, this expression does not represent the aggregate equilibrium growth rate in an economy with heterogeneous cohorts. In that case, the equilibrium growth rate can be either higher or lower than the rate at which household consumption changes with age, depending upon the economy's demographic structure.

Integrating equation (6), we express the agent's consumption level at age *z* (relative to that at age *x*) in the form

$$c_t(z) = c_{t-(z-x)}(x) \cdot e^{\frac{r-\rho}{1-\varepsilon} \cdot (z-x)}. \tag{7}$$

To express this in terms of the agent's financial resources, we proceed as follows. First, integrate equation (5) to obtain

$$R(z,x) = e^{-r(z-x)} \cdot \frac{S(z)}{S(x)}. \tag{5'}$$

Second, integrate (1b') forward at age *z* and use the transversality condition, (3c), to obtain the agent's inter-temporal budget constraint applicable from age *x* as

$$\int_{z=x}^{z=\omega} R(z,x) \cdot c_t(z)\, dz = f_t(x) + \int_{z=x}^{z=\omega} R(z,x) \cdot w_t(z) \cdot L(z) \cdot dz. \tag{8}$$

Finally, substituting for (7) and (5') into (8), the agent's consumption at age *x* can be expressed as

$$c_t(x) = m(x) \cdot v_t(x) \tag{9a}$$

where $v_t(x)$ is the "all-inclusive" wealth of a household of age *x* at time *t*, defined as

[12] See e.g. Romer (1986). But it also describes the equilibrium consumption growth rate in the two-sector Lucas (1988) model, as developed by Bond, Wang, and Yip (1996).

$$v_t(x) \equiv f_t(x) + h_t(x) = f_t(x) + \int_{z=x}^{z=\omega} e^{-r\cdot(z-x)} \cdot \left(\frac{S(z)}{S(x)}\right) \cdot w_t(z) \cdot L(z) \cdot dz \qquad (9b)$$

and $m(x)$ denotes the household marginal (and average) propensity to consume out of current all-inclusive wealth at age x, defined by[13]

$$m(x) = \left[\int_{z=x}^{z=\omega} e^{\frac{\varepsilon \cdot r - \rho}{1-\varepsilon}\cdot(z-x)} \cdot \left(\frac{S(z)}{S(x)}\right) \cdot dz\right]^{-1}. \qquad (9c)$$

That is, at time t a household of age x spends a fraction $m(x)$ of its all-inclusive wealth $v_t(x)$, which is equal to its financial assets, $f_t(x)$, plus its human wealth, $h_t(x)$, where human wealth is equal to the present value of the household's expected future labor income. Note that both wealth $v_t(x)$ and the marginal propensity to spend out of wealth depend on the agent's age and expected mortality. Henceforth, we will refer to a household's all-inclusive wealth as simply its "wealth", with $f_t(x)$ distinguished as financial wealth where it is needed for clarification.

We assume that the productivity of labor increases over calendar time at a constant rate g (to be determined in equilibrium as the "economic growth rate".) This market wage is economy-wide and common to all households, regardless of their birth dates. Thus the market wage at time t can be expressed $w_t = w_{t-x} \cdot e^{g\cdot x}$ where w_{t-x} is the wage rate prevailing in the economy at the time a household of age x is born. Substituting into $h_t(x)$, defined in (9b), the human wealth of a household of age x at time t is given by

$$h_t(x) = w_t \cdot p(x) \qquad (10a)$$

where $$p(x) = \int_{z=x}^{z=\omega} e^{-(r-g)\cdot(z-x)} \cdot \left(\frac{S(z)}{S(x)}\right) \cdot L(z) \cdot dz\,. \qquad (10b)$$

Written in this way, we see that the human wealth of an agent equals the current wage rate scaled by a present value factor $p(x)$ (which is independent of t), which reflects the discounted future labor supply, adjusted for the rate of productivity growth and the agent's probability of survival. From

[13] Note that for the infinitely-lived household with logarithmic utility ($\omega \to \infty$, $\varepsilon \to 0$), the marginal propensity to consume wealth in (9c) is just the familiar constant ρ.

equation (10), the initial human wealth of a household of age x at time t is $h_{t-x}(0) = w_{t-x} \cdot p(0)$, where $p(0)$ is found by setting $x = 0$ in equation (10b).

If every household begins with no financial wealth (no inheritance), the overall wealth at birth of a household currently aged x consists entirely of its initial human wealth, or $v_{t-x}(0) = h_{t-x}(0) = w_{t-x} \cdot p(0)$. Combining equations (9a) and (7), we can write $v_t(x) = c_t(x) \cdot m(x)^{-1} = c_{t-x}(0) \cdot e^{\left(\frac{r-\rho}{1-\varepsilon}\right)\cdot x} \cdot m(x)^{-1}$. With no initial financial wealth, (9a) further implies that for this cohort $c_{t-x}(0) = m(0) \cdot p(0) \cdot w_{t-x}$ which when combined with the previous equation yields

$$v_t(x) = \frac{m(0)}{m(x)} \cdot p(0) \cdot w_t \cdot e^{\left(\frac{r-\rho}{1-\varepsilon}-g\right)\cdot x}. \qquad (11)$$

where $m(0)$, the household's marginal propensity to consume wealth at birth, is obtained by setting $x = 0$ in equation (9c).

3. The Aggregate Economy

To derive the aggregate economy we need to describe its demographic structure. Let B_{t-x} denote the size of the population cohort born at time *t-x*. Given the survivorship function, $S(x)$, the size of that cohort (now of age x) at time t is $N_t(x) = B_{t-x} \cdot S(x)$. We restrict our analysis to economies for which birth cohorts and population grow at a constant rate over time. Assuming that birth cohorts grow at rate n over time, $B_t = B_{t-x} \cdot e^{n\cdot x}$ and we may express $N_t(x)$ in terms of the size of the current birth cohort as $N_t(x) = B_t \cdot e^{-n\cdot x} \cdot S(x)$. Aggregating over all cohorts, the total population size at time t is $N_t = \int_{x=0}^{x=\omega} N_t(x) \cdot dx = B_t \cdot \int_{x=0}^{x=\omega} e^{-n\cdot x} \cdot S(x) \cdot dx$. Given the time-invariance of the survivorship function, the total population also grows at rate n.

The number of deaths at time t of persons of age x is $D_t(x) = -\left(S'(x)/S(x)\right) \cdot N_t(x)$ so that the total number of deaths equals $D_t = -\int_{x=0}^{x=\omega} S'(x) \cdot B_{t-x} \cdot dx$. Let *b(x)* denote the fertility or birth rate by individuals of age *x*, so the number of births at time *t* to persons of age *x* is $b(x) \cdot N_t(x)$. Then, the total number of births equals $B_t = \int_{x=0}^{x=\omega} b(x) \cdot S(x) \cdot B_{t-x} \cdot dx$. Since $n \cdot N_t = B_t - D_t$ and

$B_{t-x} = B_t \cdot e^{-n \cdot x}$, we can substitute, rearrange, and integrate by parts to obtain the following constraint on the chosen demographic functions:

$$\int_{x=0}^{x=\omega} b(x) \cdot S(x) \cdot e^{-n \cdot x} dx = 1. \tag{12}$$

Equation (12) represents a demographic "adding-up" constraint on our choices of the population growth rate and the survivorship and fertility functions.[14] Whenever we compute the effects of demographic changes on the macroeconomic equilibrium,(as in Section 5 where compute the impact on the growth rate), this constraint must be taken into account.

3.1 The Aggregate Household Sector

We now use the demographic structure to obtain the key aggregate economic variables: the aggregate labor supply, labor income, consumption, and financial wealth.

(i) **Aggregate labor supply**, L_t, at time t is obtained by summing the labor supply across all cohorts. That is,

$$L_t = \int_{x=0}^{x=\omega} N_t(x) L(x) dx = B_t \int_{x=0}^{x=\omega} e^{-n \cdot x} \cdot S(x) L(x) dx = B_t \cdot \Sigma^L \tag{13a}$$

where $$\Sigma^L \equiv \int_{x=0}^{x=\omega} e^{-n \cdot x} \cdot S(x) \cdot L(x) \cdot dx. \tag{13b}$$

In equation (13a), the time independent coefficient, Σ^L is equal to the ratio of the labor supply to the size of the birth cohort time at any time t.[15]

(ii) **Aggregate labor income** earned at time t is obtained by aggregating over all cohorts (and taking account of their respective survival rates) to get $Y_t^L = \int_{x=0}^{x=\omega} N_t(x) \cdot L(x) \cdot w(x) \cdot dx$. Since all agents are paid the same prevailing wage at time t, $w(x) = w_t$, we can write

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[14] For example, if b is independent of age and $S(x)$ is exponentially declining (a constant mortality hazard rate of θ), equation (12) implies $b = \theta + n$. In general, the demographic constraint is more complicated.

[15] Whenever an aggregate variable is obtained by summing over households of all ages in the population, we denote the resulting age and time-independent coefficient by Σ superscripted by the corresponding variable.

$$Y_t^L = w_t \cdot L_t = w_t \cdot B_t \cdot \Sigma^L \tag{13c}$$

From (13c) we see that aggregate labor income Y_t^L grows at rate $n+g$, the sum of the growth rates of the labor supply L_t and labor productivity, as reflected in the growth rate of the wage rate.

(iii) **Aggregate consumption** at time t is $C_t = \int_{x=0}^{x=\omega} N_t(x) \cdot c_t(x) \cdot dx$. Using equations (9a) and (11), we obtain

$$C_t = Y_t^L \cdot \frac{\Sigma^C}{\Sigma^L} \tag{14a}$$

where
$$\Sigma^C = m(0) \cdot p(0) \cdot \int_{x=0}^{x=\omega} e^{\left(\frac{r-\rho}{1-\varepsilon}-g-n\right)\cdot x} S(x) \cdot dx. \tag{14b}$$

In equation (14a), the time-independent coefficient Σ^C / Σ^L is the ratio of aggregate consumption to aggregate labor income.

(iv) **Aggregate financial wealth** at time t is $F_t \equiv \int_{x=0}^{x=\omega} N_t(x) \cdot f(x) \cdot dx$. Aggregate financial wealth is also equal to **aggregate all-inclusive wealth** V_t minus **aggregate human wealth** H_t, where $V_t \equiv \int_{x=0}^{x=\omega} N_t(x) v_t(x) \cdot dx = B_t \cdot w_t \cdot p(0) \cdot m(0) \cdot \int_{x=0}^{x=\omega} e^{\left(\frac{r-\rho}{1-\varepsilon}-g-n\right)\cdot x} \cdot S(x) \cdot m(x)^{-1} \cdot dx$ and $H_t \equiv \int_{x=0}^{x=\omega} N_t(x) \cdot h_t(x) \cdot dx = w_t B_t \int_{x=0}^{x=\omega} e^{-n \cdot x} \cdot S(x) \cdot p(x) \cdot dx$. Substituting equation (9c) for $m(x)^{-1}$ and equation 10(b) for $p(x)$ we can carry out the integration to obtain[16]

$$F_t = V_t - H_t = \left[\frac{\Sigma^C - \Sigma^L}{\Sigma^L \cdot (r-g-n)} \right] \cdot Y_t^L. \tag{15}$$

In equation (15), the time independent coefficient $\left[(\Sigma^C - \Sigma^L) / \Sigma^L \cdot (r-g-n) \right]$ is the ratio of the aggregate financial capital to aggregate labor income. We assume financial wealth in the economy is positive, so we assume this coefficient is positive.[17] Combining (14a) with (15), we see that

[16] Substituting (9c) and (10b) yields double integrals which can be rearranged using the equality $\int_{x=0}^{x=\omega} \int_{z=x}^{z=\omega} f(x,z) \cdot dz \cdot dx = \int_{z=0}^{z=\omega} \int_{x=0}^{x=z} f(x,z) \cdot dx \cdot dz$. This equality obtains because the integrand $f(x,y)$ is integrated over the same triangular area domain. Also, we assume $r \neq g+n$, which is proved to be the case in the Appendix.

[17] Technically, an individual household could hold negative annuity wealth. This would be equivalent to borrowing and buying life-insurance to retire the debt in case of death. We assume that aggregate household annuity wealth is positive.

$$C_t = \left[\frac{\Sigma^C \cdot (r-g-n)}{\Sigma^C - \Sigma^L} \right] \cdot F_t. \tag{16}$$

In equation (16), the time independent coefficient $\left[\Sigma^C \cdot (r-g-n) / (\Sigma^C - \Sigma^L) \right]$ measures the aggregate marginal and average propensity to consume out of financial wealth.

The coefficients $\left[m(0), p(0), \Sigma^C, \Sigma^L, \right]$ in equations (11) through (16) depend upon (i) the taste parameters ε and ρ, (ii) the (constant) values for the population growth rate *n*, the productivity growth rate *g*, and the rate of return on capital *r*, and (iii) the forms of the demographic functions. Consequently, the aggregate variables $\left[C_t, F_t, Y_t^L \right]$ grow at the sum of the growth rates of the labor supply and labor productivity $n+g$.

3.2 The Aggregate Production Sector

In deriving the behavior of the household, we have assumed that the rate of return on capital and the growth rate of labor productivity are constant over time, and along with the prevailing wage rate, are exogenous to the household. To complete the model and determine the equilibrium, the values of $w_t, r,$ and *g* are derived. These values depend on the underlying production technology. In the case of the Romer production function, with its implied constant productivity of capital (AK) technology, *r* and *g* will indeed be constant, consistent with our maintained assumptions.[18]

We assume that there are L_t identical firms, and each hires one unit of labor and k_t units of capital. Firm output net of capital replacement, q_t, is produced in accordance with a Romer-type (1986) Cobb-Douglas production function[19]

$$q_t = A \cdot k_t^a \cdot \left(\frac{K_t}{L_t} \right)^{1-a} - \delta \cdot k_t \tag{17}$$

where *A* is the total factor productivity term, k_t denotes the firm's capital stock, δ is the depreciation rate, and K_t/L_t is the economy-wide capital-labor ratio. This last term provides the production externality that ensures that the equilibrium productivity of capital remains constant,

[18] If the underlying production function is neoclassical, these quantities would be time varying and this would need to be taken into account by the household sector in its decision-making process.

[19] While Romer (1986) specified the production function as Cobb-Douglas, the crucial properties we obtain apply to any linearly homogeneous production function of the form $f(k_t, K_t/L_t)$.

thereby enabling the economy to sustain a constant equilibrium growth rate.

Assuming each firm is small enough to ignore its own impact on the economy-wide values of K_t and L_t, and because firms are identical, in equilibrium $k_t = K_t/L_t$ and the equilibrium rate of return on capital is given by its marginal product

$$r = \alpha \cdot A - \delta \tag{18a}$$

which is constant over time. Further, aggregate output $Q_t = L_t \cdot q_t = (A - \delta) \cdot K_t$, while the equilibrium wage at time *t* is given by

$$w_t = q_t - (\alpha \cdot A - \delta) \cdot k_t = (1-\alpha) \cdot A \cdot \frac{K_t}{L_t}. \tag{18b}$$

Financial capital is the claim on physical capital in the economy so $F_t = K_t$, which grows at rate $g+n$ while L_t grows at the population growth rate *n*. Thus, the wage rate w_t grows at the constant rate g over time, validating the assumption we imposed in the household sector. The ratio of the factor income shares in production is equal to:

$$\frac{r \cdot K_t}{w_t \cdot L_t} = \frac{\alpha \cdot A - \delta}{(1-\alpha) \cdot A}. \tag{18c}$$

Using $dK_t/dt \equiv Y_t^L + r \cdot K_t - C_t$, we can express the aggregate goods market clearing condition in the economy as

$$\frac{dK_t}{dt} = (A - \delta)K_t - C_t. \tag{19}$$

The production side of the economy is fully described by equations (17) to (19).

3.3 Closing the System

The final step in determining the equilibrium growth rate g is to require that the aggregate household variables be consistent with the aggregate production variables. In the conventional Romer model, the equilibrium growth rate is obtained directly from the goods market equilibrium condition

$$\frac{\dot{K}_t}{K_t} = \frac{Q_t - C_t}{K_t} = r + (1-\alpha)\cdot A - \frac{C_t}{K_t} \tag{19'}$$

and substituting for C_t/K_t. The same procedure is performed here, although the resulting equation is nonlinear in the growth rate, raising the issue of multiple solutions. Substituting $F_t = K_t$, $\dot{K}_t/K_t = g + n$ and using equation (16), we write (19') as[20]

$$g = r - n + A(1-\alpha)\left[1 - \frac{\Sigma^C}{\Sigma^L}\right]. \tag{20}$$

Also, the saving rate, $\sigma \equiv \frac{Y_t - C_t}{Y_t}$ can be expressed as $\sigma = 1 - \left[\frac{(1-\alpha)\cdot A}{A-\delta}\right]\cdot\frac{\Sigma^C}{\Sigma^L}$, thus we can write

$$g = (A-\delta)\cdot\sigma - n. \tag{20'}$$

Equation (20) can be expressed in terms of the underlying demographic characteristics as

$$g = r - n + A(1-\alpha)\left[1 - \left(\frac{\int\limits_{x=0}^{x=\omega} e^{-(r-g)\cdot x}\cdot S(x)\cdot L(x)\cdot dx}{\int\limits_{x=0}^{x=\omega} e^{-n\cdot x}\cdot S(x)\cdot L(x)\cdot dx}\right)\left(\frac{\int\limits_{x=0}^{x=\omega} e^{\left(\frac{r-\rho}{1-\varepsilon}-g-n\right)\cdot x} S(x)\cdot dx}{\int\limits_{x=0}^{x=\omega} e^{\frac{\varepsilon\cdot r-\rho}{1-\varepsilon}\cdot x}\cdot S(x)\cdot dx}\right)\right] \tag{20''}$$

From (20''), it is apparent that $g = r - n$ is a solution for any arbitrary functions $S(x)$ and $L(x)$. In the Appendix we show that this solution is incompatible with the household's inter-temporal budget constraint because it violates the transversality conditions. Therefore, we rule out this degenerate solution. In Section 5 below we numerically solve (20'') for its proper solutions given different specifications of the demographic functions.

3.4 Underlying Dynamics

The equilibrium we have been describing assumes that the economy is always on its balanced growth path. Given the fact that the underlying technology is based on the one-sector Romer technology, for which the infinite-lived representative agent economy has this property, this is a

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[20] Equation (20) can also be obtained by setting the ratio of factor income shares in the aggregate household sector equal to the relative factor income shares on the production side. That is, $r\cdot F_t/Y_t^L = (\alpha\cdot A-\delta)(1-\alpha)^{-1}\cdot A^{-1}$ and substitute equation (15) into the left hand side to obtain equation (20) directly.

natural assumption. Nevertheless, it requires further discussion and justification in the context of this more general demographic structure.

In a recent paper, Mierau and Turnovsky, (2011) have embedded a general demographic structure in a conventional Ramsey growth model. They show that, as in the standard model, the aggregate macroeconomic equilibrium can be summarized by two dynamic equations in (i) per capita capital stock and (ii) per capita consumption. The first equation is the usual product market clearing condition, while the second is the Euler equation, determining the consumption growth rate, which now is modified by the inclusion of what they call the "demographic turnover term" and they denote by $\Phi(t)$. This term summarizes the demographic structure, and is the channel whereby all demographic structures impinge on the dynamics. In the case of the Blanchard model, the dynamics of $\Phi(t)$ is very straightforward and the neoclassical production function preserves the saddlepoint structure associated with the standard Ramsey technology. However, for more realistic demographic structures, $\Phi(t)$ is very complex, reflecting variations in the marginal propensity to consumption over the life cycle and therefore across cohorts. The finite lifespan typically means that the overall equilibrium involves the analysis of mixed differential-difference equations, which present a huge computational challenge. Indeed, as d'Albis and Augeraud-Véron (2009) emphasize that the characterization of the dynamics in terms of a mixed-differential difference equation is an essential generic feature of continuous-time overlapping generations models, with the Blanchard model being one of the few exceptions.[21]

But despite the intractability of the general system set out by Mierau and Turnovsky, it does provide some relevant insights. First, for the Romer technology, the equilibrium dynamics does simplify substantially. One can easily establish that the Blanchard model is always on its balanced growth path, as is the infinitely-lived representative agent economy. Furthermore, one can show that the same is true for the Samuelson model. The intermediate de Moivre case remains intractable. However, the fact that the two polar cases [Blanchard and Samuelson] that bracket this intermediate

[21] Mierau and Turnovsky (2011) lay out the equilibrium dynamics for the Boucekkine et al. (2002) demographic structure. It is a fifth order mixed differential-difference system and is almost certainly intractable.

case are always on their respective balanced growth paths, provides some support for focusing on the balanced growth in all cases, as we are doing.[22]

4. Parameterizing the Demographic Functions

In order to evaluate the coefficients $\left[m(0), p(0), \Sigma^C, \Sigma^L\right]$ appearing in the expressions that determine the equilibrium economic growth rate, we parameterize the demographic functions $b(x)$, $S(x)$ and $L(x)$ that describe how fertility, survivorship and labor supply vary with a household's age. We assume these functions are exogenous, although in reality they may reflect household choices, particularly the labor supply and fertility functions. In general, a household's fertility, survivorship and labor supply decline with age.

While we take the demographic structure to be exogenous, there is a growing literature endogenizing this aspect, and in the process appealing to different mechanisms. For example, using the Barro-Becker (1989) model, Manuelli and Seshadri (2009) show how fertility and mortality differences across countries can be accounted for by differences in productivity and in labor income tax rates. Because of the contrasting ways these two variables are related to the growth rate, this enables them to show how the relationship between demographic changes and growth depend upon the source of the demographic change.[23] In contrast, Soares (2005) studies the relationship between the declining mortality rates at birth, leading to reductions in fertility, and followed by increases in the rate of human capital accumulation. Another approach is taken by Doepke (2004), who emphasizes the importance of laws regulating child labor in accounting for the demographic transition from high to low fertility accompanying the process of industrialization.

Arguably the most important element is the survivorship function. The standard infinite-lived agent growth model has no mortality, of course, while the Blanchard model assumes that survivorship declines exponentially. While this is tractable and easy to pair with exponential growth models, the assumption of a constant mortality hazard rate (so-called "perpetual youth") is at odds

[22] We should add that these comments apply to standard models that abstract from retirement. The introduction of a finite working life that in general does not coincide with the finite lifetime makes the formal analysis of the transitional dynamics even more intractable in the general case.

[23] We deal with this issue partially in Tables 2 and 3 where we contrast the effects of population growth due to exogenous changes in fertility with those due to exogenous changes in longevity on the growth rate and savings.

with the facts of human mortality. Some authors have analyzed overlapping generations using more realistic survivorship functions. For example, Boucekkine, de la Croix, and Licandro (2002) develop a growth model using an interesting two-parameter exponential survivorship function that exhibits a mortality hazard rate that increases with age.[24] However, growth in their model is based on human capital accumulation and the model assumes linear utility, which makes it difficult to compare to standard growth models. Heijdra and Romp (2008) use the Gompertz (1825) exponential mortality hazard function in a small open-economy overlapping generations model. Faruqee (2003) approximates the Gompertz function with an estimated hyperbolic function, which he introduces into the Blanchard (1985) model to examine the effect of a more plausible demographic structure on issues pertaining to the financing of government debt and Ricardian equivalence. Neither of these studies includes a production sector, and both therefore abstract from capital accumulation and economic growth. The Gompertz survivorship function, while tracking human mortality well, is intractable for our purposes.[25]

One hundred years before Gompertz, de Moivre (1725) observed about life expectancy that "…the number of lives existing at any age is proportional to the number of years intercepted between the age given and the extremity of old age". This suggests a simple two parameter survivorship function that can be written in the form

$$S(x) = \left(1 - \frac{x}{\omega}\right)^{\theta \cdot \omega - 1} \tag{21}$$

where ω is the maximum age to which a household can live and $1 \le \omega \cdot \theta$ (that is, $1/\theta \le \omega$).[26] The de Moivre function is reasonably tractable yet fits the main features of modern human mortality quite well. Most importantly, from our standpoint, the Blanchard "perpetual youth" and the

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[24] Boucekkine *et al* specify $S(x) = (e^{-\beta \cdot x} - \alpha)/(1-\alpha)$ for $0 \le x \le -\log(\alpha)/\beta$.

[25] The Gompertz log linear mortality hazard function implies a two parameter survivorship function of the Gumbel extreme value form. Specifically, $S(x) = e^{\alpha(1-e^{\beta x})/\beta}$ where $\alpha, \beta > 0$ are parameters. The exponential survivorship function is a limiting case of the Gompertz survivorship function when $\beta \to 0$.

[26] Originally, de Moivre proposed a linear function, where $\theta\omega = 2$, which is one of the special cases we shall compute numerically in Section 5. The de Moivre and the Gompertz survivorship functions are nested in the function $S(x) = \exp\left\{((\theta\omega - 1)/(1-\beta\omega)) \cdot \left[(1 - x/\omega)^{1-\beta\omega} - 1\right]\right\}$ which converges to the de Moivre as $\beta\omega \to 1$ and the Gompertz as $\omega \to \infty$ (see Kohler and Kohler (2000).)

Samuelson “one-hoss shay” models emerge as limiting specifications of a more general model. In particular, the limit of expression (21) as $\omega \to \infty$ is Blanchard’s single parameter exponential survivorship function $S(z)=e^{-\theta\cdot x}$.[27] The Samuelson specification is obtained by setting $\theta\cdot\omega=1$. In this case, the household survives with certainty until age ω, at which time it dies with certainty. Both specifications converge smoothly to the representative (infinitely–lived) agent case, common in the growth literature, by letting either $\theta\to 0$ or $\omega\to\infty$.

More generally, if $\theta\cdot\omega<(>)2$, the survivorship function is strictly concave (convex) as shown in Figure 1. This figure illustrates the two polar specifications, Samuelson and Blanchard, together with two particular intermediate cases. In the first intermediate case, we assume that life expectancy at birth is equal to half the longest possible life span which implies that survivorship declines linearly. We denote this specification as “Intermediate L”. In the second intermediate case, we assume that life expectancy at birth is equal to 2/3 of the longest possible life span[28], so the survivorship function takes a square-root form. We denote this specification as “Intermediate-SQ”. In this case, the survivorship function is strictly concave and approximates an actual survivorship function for a developed economy.[29]

Similarly, we parameterize the household labor supply using the de Moivre function

$$L(x)=\left(1-\frac{x}{\ell}\right)^{\lambda\cdot\ell-1}. \tag{22}$$

This labor supply function, which describes how a household reduces its labor supply as it ages, has the same possible shapes as the survivorship function in Figure 1. Again, there are two limiting cases: i) an exponentially declining labor supply function when $\lim_{\ell\to\infty} L(x)=e^{-\lambda\cdot x}$ and ii) a one-hoss shay labor supply function as the limit as $\lambda\cdot\ell\to 1$, which implies that $L(x)=1$ for $0\le x<\ell$ and $L(x)=0$ otherwise.[30] The first limiting case is similar to the generalization of the Blanchard model

[27] The limiting cases as $\omega\to\infty$ can be evaluated using $\lim_{\omega\to\infty}(1-x/\omega)^{\theta\cdot\omega}=e^{-\theta\cdot x}$

[28] Life expectancy at birth is about 80 years in developed economies, and the documented longest life is 122 years, so we treat this as the most realistic case.

[29] Actual survivorship functions are concave except at extreme old age.

[30] A useful extension would be to derive the labor supply function from underlying behavioral considerations.

developed by Blanchard and Fischer (1989).[31] The household retires "gradually", reducing its labor supply at rate $\lambda \cdot e^{-\lambda \cdot x}$ at age *x*. The second limiting case is similar to the Samuelson assumption that a household supplies one unit of labor until it reaches a given retirement age $\ell \le \omega$, at which time it fully retires and provides zero units thereafter. In our Samuelson specification, the household lives and works for an interval of length ℓ and then retires and supplies no labor over the remaining time interval $\omega - \ell$.[32]

4.2 The Parameterized Growth Model

Equations (23) summarize the aggregate economy in its parameterized form.

$$\frac{\dot{K}}{K} = g + n = r + A \cdot (1-\alpha)\left[1 - \frac{\Sigma^C}{\Sigma^L}\right] \tag{23a}$$

$$\Sigma^L = \int_{x=0}^{x=\omega} e^{-n \cdot x} \cdot \left(1 - \frac{x}{\omega}\right)^{\theta \cdot \omega - 1} \cdot \left(1 - \frac{x}{\ell}\right)^{\lambda \cdot \ell - 1} \cdot dx \tag{23b}$$

$$\Sigma^C = m(0) \cdot p(0) \cdot \int_{x=0}^{x=\omega} e^{\left(\frac{r-\rho}{1-\varepsilon} - g - n\right) \cdot x} \cdot \left(1 - \frac{x}{\omega}\right)^{\theta \cdot \omega - 1} \cdot dx \tag{23c}$$

$$m(0) = \left(\int_{x=0}^{x=\omega} e^{\frac{\varepsilon \cdot r - \rho}{1-\varepsilon} \cdot x} \cdot \left(1 - \frac{x}{\omega}\right)^{\theta \cdot \omega - 1} \cdot dx\right)^{-1} \tag{23d}$$

$$p(0) = \int_{x=0}^{x=\omega} e^{(g-r) \cdot x} \cdot \left(1 - \frac{x}{\omega}\right)^{\theta \cdot \omega - 1} \cdot \left(1 - \frac{x}{\ell}\right)^{\lambda \cdot \ell - 1} \cdot dx \tag{23e}$$

$$r = \alpha \cdot A - \delta \tag{23f}$$

$$\sigma \equiv \frac{Y_t - C_t}{Y_t} = 1 - \left[\frac{(1-\alpha) \cdot A}{A - \delta}\right] \cdot \frac{\Sigma^C}{\Sigma^L} \tag{23g}$$

The aggregate equilibrium growth rate, *g*, can be obtained by substituting (23b)-(23f) into the goods market clearing condition, (23a), and solving for *g*, while the aggregate savings rate in the economy,

[31] In his original paper, Blanchard assumed that labor earnings are constant over time, while Blanchard and Fischer (1989) assumed that they declined with age. Since labor productivity is grows at rate *g* with time, labor earnings change with age at rate *g-λ*.

[32] However, we do not require that the working and retirement periods be of equal length, as is the case in some discrete time Samuelson models.

denoted by σ, is obtained from (23g). Using the equilibrium g, the values of the aggregate economic variables $\left[Y_t^L, C_t, F_t = K_t, w_t\right]$ are determined using the parameterized versions of equations (13c), (14), (15) and (18b) respectively.

In general, we are unable to obtain closed form solutions for the macroeconomic equilibrium described by (23), except for the polar cases of the Blanchard "perpetual youth" and the Samuelson "one-hoss shay" specifications. The solutions for these two classic models are found in Appendix A.2. There we also show that if the population is constant ($n = 0$) and households are infinitely lived (that is, $(\lambda, \theta) \to 0$ in the Blanchard specification and $(l, \omega) \to \infty$ in the Samuelson specification), the equilibrium growth rate reduces to the standard Romer result, $g = (r - \rho)/(1 - \varepsilon)$.

5. Numerical Computations of Equilibrium Economic Growth Rates

To proceed further we compute equilibrium growth rates for specified values of the parameters.[33] We begin by establishing a benchmark specification which we use to compare the growth rates in the Blanchard, Intermediate, and Samuelson specifications of the general OLG model, and we compare these OLG growth rates to that of the standard representative agent growth model. Because the production and preference characteristics of the economy are standard and well documented in the literature, we maintain the following values throughout our analysis. On the production side we assume $A = 0.35, \alpha = 1/3, \delta = 0.05$, which implies a return to capital $r = 0.067$. For preference parameters, we assume $\rho = 0.03$ and $\varepsilon = -1.5$ (that is, an inter-temporal elasticity of substitution equal to 0.4).[34] From equation (6), this implies a growth rate for the standard infinitely-lived representative agent model of 1.47%.

In contrast, we extensively vary the demographic assumptions, and compute the growth rate for four specifications of the de Moivre survivorship function: the Blanchard (exponential) case, the

[33] We use the FindRoot utility in *Mathematica* to solve equations (23) simultaneously. The *Mathematica* code is available from the authors on request. In the Blanchard and Samuelson cases, the program finds the roots of explicit equations shown in Appendix 2. In the intermediate cases, the program finds roots of integral equations, which proves to be quite challenging. Although multiple roots are possible, we can eliminate roots with negative financial wealth. In all cases, only a single root is found in the relevant range.

[34] The only point to note is that ρ, being a pure rate of time preference, is somewhat smaller than the conventional value for the representative agent model ($\rho = 0.04$). This is because the latter implicitly discount for mortality factors, which we are explicitly incorporating in our analysis.

linear case (Intermediate L), the concave case (Intermediate SQ), and the Samuelson one-hoss shay case. The shape of the survivorship function in each case is shown in Figure 1. Except when we specify "no retirement", we assume a shape for the labor supply function that corresponds to that of the survivorship function. We allow the population growth rate n to vary between 0 (stationary population) and 3%, which we characterize as a high population growth rate, with a benchmark value for population growth equal to 1.5%. We also vary life expectancy in the economy. A household's life expectancy at the time it enters the economy, denoted χ, is given by

$$\chi = \int_{x=0}^{x=\omega} S(x) \cdot dx = \int_{x=0}^{x=\omega} \left(1 - \frac{x}{\omega}\right)^{\theta \cdot \omega - 1} \cdot dx. \tag{24}$$

Our benchmark life expectancy is 60 years, but we vary life expectancy in our simulations between 40 years and 90 years.[35]

We mentioned earlier that in, any economy, fertility, survivorship and the population growth rate must satisfy the demographic "adding up" constraint specified in equation (12). In our simulations, we assume (unrealistically) that the birth rate does not depend on age, so $b(x) = b$.[36] Assuming an age-independent birth rate and a de Moivre survivorship function, we can write equation (12) in parameterized form as:

$$b = \left(\int_{x=0}^{x=\omega} \left(1 - \frac{x}{\omega}\right)^{\theta \cdot \omega - 1} \cdot e^{-n \cdot x} \cdot dx \right)^{-1}. \tag{25}$$

Equation (25) constrains our choice of the four demographic parameters $[b, n, \theta, \omega]$ in the numerical comparisons below. In some cases, we vary the population growth rate and/or life expectancy, assuming that the birth rate adjusts residually to satisfy equation (25). In other cases, we hold the birth rate constant so that, for example, a change in life expectancy results in a change in the population growth rate.

[35] We assume the household enters the economy as an adult of 20 years so that life expectancy at birth is approximately 80 years. Since childhood is ignored in this model, we refer to the time the household enters the economy and the time of its birth interchangeably.

[36] When comparing b to actual values, it should be kept in mind that measured birth rates are expressed as a percentage of the whole population. In this paper we consider only households over age 20, which is about 2/3 of the total population, so b is about 50% higher than measured birth rates. Also, more generally, we can specify a fertility function of a de Moivre form, so that fertility declines with age. In addition to being more realistic, varying the parameters of the function would allow us to examine the phenomenon of delayed fertility on the economic growth rate.

Finally, we vary the fraction of its expected life that the household expects to work. This requires us to vary the parameters of the household's survivorship and labor supply functions so as to satisfy

$$\psi = \int_{x=0}^{x=\ell} \left(1-\frac{x}{\ell}\right)^{\lambda\cdot\ell-1} \cdot \left(1-\frac{x}{\omega}\right)^{\theta\cdot\omega-1} \cdot dx \cdot \left(\int_{x=0}^{x=\omega} \left(1-\frac{x}{\omega}\right)^{\theta\cdot\omega-1} \cdot dx \right)^{-1} \tag{26}$$

where ψ denotes the ratio of the expected number of years a household works to its life expectancy.[37] Our benchmark value for ψ is 2/3 (the expected working life of a household is equal to two-thirds of its expected life), and we vary the value between one-half and three-quarters. For example, in the Samuelson specification, our benchmark simply requires that we set $l = 40, \omega = 60$, while in the Blanchard specification, it requires that we set $\theta = 1/60$ and $\lambda = \theta/2 = 1/120$. The values must be computed in the intermediate cases.

5.1 The Overall Effect of Demographic Model Structure

Table 1 shows the overall effect of introducing demographic features into the Romer (1986) economy with constant population. The first row shows the case where households have life expectancy equal to 60 and work their entire lives. In this case, the introduction of mortality reduces the equilibrium saving and economic growth rates, and the reduction is greater if the survivorship function is more concave. Our standard production and preference parameters imply that households increase their consumption as they age by 1.47% per year. In the infinitely-lived, representative agent model (designated the "Romer case" in the table), this is the equilibrium growth rate for the aggregate economy and for household consumption, saving, and wealth. However, with mortality, desired household wealth does not increase at this rate because of the household's finite horizon, so the equilibrium growth rate for the economy (and the household's labor income) must be less than the rate at which household consumption rises with age. The more concave the survivorship function (the more certain the household reaches old age), the lower is the equilibrium growth rate. When the survivorship function is sufficiently concave, households dis-save in the latter part of their lives

[37] Where mortality is uncertain, the calculation of the expected working life includes the fact that the household may die while working.

despite a desire for rising consumption. With life expectancy equal to 60 years in Table 1, the equilibrium growth rate is driven to zero in the Samuelson polar case. However, the equilibrium growth rate is positive in the Samuelson case if life expectancy is sufficiently longer. Indeed, when households do not retire, the equilibrium growth rates in all of the mortality cases approach that of the Romer model (1.47%) as life expectancy increases without bound. (As proven for the Blanchard and Samuelson specifications in the Appendix, and can be confirmed numerically.) The important implication of this analysis is that, holding life expectancy constant and without retirement, the greater certainty of reaching old age (more concave survivorship functions) found in developed economies would reduce equilibrium growth rates.[38]

Another important demographic phenomenon, at least in developed economies, is the fact that households reduce their labor supply as they age, a phenomenon we call retirement.[39] Retirement introduces an additional incentive for households to save when they are young and dis-save when they are old. In the second row of Table 1, we assume that households expect to work two-thirds of their expected lives (that is, we set ψ equal to 2/3). This exogenously introduced retirement increases the aggregate saving and growth rates in all mortality cases, and the increase is greater the more concave the survivorship function. As mentioned, a more concave survivorship function means households have greater certainty about time of death. When households do not retire, their incentive to save is reduced if they are more certain of receiving growing labor income as they age. The lower saving, in turn, would eliminate the source of growth, so the equilibrium growth rate is reduced. However, when households retire, a more concave survivorship function implies greater certainty of reaching an old age with higher consumption *and* no earnings. Because the households exit the labor force as they age, anticipated wage growth does not reduce their incentive to save as it does in the no retirement case, so higher equilibrium growth rates exist.

As seen in Table 1, the growth rate is 1.08 percentage points higher in the Samuelson specification (the most concave survivorship function) than in the Romer case, while the saving rate is 3.62 percentage points higher. By comparison, the Blanchard rates are only modestly different

[38] Of course, greater certainty of reaching old age has been accompanied with increased life expectancy and longer retirements in developed economies.
[39] In this paper, we assume households must finance their own retirement and ignore social security.

from the Romer case, where with infinitely-lived households, retirement has no effect. Not surprisingly, the saving and growth rates for the intermediate specifications of the survivorship function lie between those of the Blanchard and Samuelson specifications. Perhaps the most realistic case is Intermediate SQ specification, where we assume the maximum possible age a household can live (ω) is 50% higher than life expectancy, which approximates observations found in developed economies. In this case, the economy grows at 1.88% and the aggregate saving rate is 6.27% for our benchmark specification.

To summarize, mortality hazard and overlapping generations reduces saving and growth rates relative to that of the representative agent model of growth, but mortality hazard coupled with a realistic retirement assumption increases saving and growth rates, particularly when the survivorship function is concave. The concavity of the survivorship function has increased significantly for developed economies over the past hundred years (see Figure 2 for the United States), because mortality hazard rates for young and middle-aged persons have dropped significantly. This trend towards reaching old age with greater certainty, when coupled with saving for retirement, increases the equilibrium growth rate in our benchmark economy.

5.2 The Effects of Population Growth and Demographic Transition

Ever since Mathus' "Essay on the Principle of Population", there has been concern that high population growth rates may impoverish a nation. Indeed, the fall in fertility over the past two centuries in what are now developed economies has coincided with these nations growing rich.[40] Today, a similar but faster "demographic transition" is taking place in some developing economies (Bongaarts, 2009). On the other hand, cross-country studies of fertility and growth have not consistently found a statistically significant association between economic growth and population growth, although correlations are typically negative for less developed economies and positive for developed economies (Kelley, 1988). The effect of population growth has also been found to depend on the source of population growth, with negative effects from higher fertility and positive effects from reduced mortality (Kelley and Schmidt, 1995).

[40] Of course, causality can go either or both ways between population growth and per capita income growth.

Table 2 shows the effects on saving and economic growth rates of higher population growth rate due to higher birth rates. Saving and economic growth rates are computed for economies where the population is stable, growing at 1.5%, and growing at 3%. In these economies, we assume that life expectancy is 60 years and households expect to work 2/3 of their expected lives. The birth rates required to support these population growth rates, as calculated from demographic constraint given by equation (25), are also reported.

As seen in the table, the effect of population growth is similar in all specifications, with the economic growth rate cut almost in half when the population growth rate is increased from zero to 3% through a rise in the birth rate. The fall in the saving and economic growth rates are proportionally greater in the Blanchard specification than in the Samuelson specification, although the differences between all four specifications are modest. In the Intermediate SQ specification, which we describe as the “most realistic”, the growth rate drops from 1.88% for a stable population to 1.11% for a population growing at 3%, while the saving rate rises from 6.27% to 13.69%. Although higher population growth decreases economic growth, it increases the saving rate in all specifications, because the proportion of younger (saving) households is increased. Although we might expect higher saving should lead to higher growth, the economic growth rate falls because the higher rate of capital formation is insufficient to offset the effect of the higher population growth on the labor supply and labor productivity.

In Table 3, we hold the birth rate constant at 3% and consider population growth rate changes that result from reduced mortality (increased life expectancy). We vary the parameters of the survivorship function to obtain the desired life expectancies using equation (24), and the population growth rates, reported in the table, are calculated using the demographic constraint (25). At our benchmark life expectancy of 60 years, and assuming a household expects to work two-thirds of its expected life, the economic growth rate ranges from 1.34% in the Blanchard specification to 1.74% in the Samuelson specification. As we vary life expectancy from 40 years to 90 years, holding the working time ratio constant, the economic growth rate is reduced modestly in all specifications. However, the variation in population growth rates is smaller in Table 3 than in Table 2. The decline in the economic growth rate *per percentage point increase* in the population growth rate is somewhat greater when population growth is induced by higher fertility than by reduced mortality. In the Intermediate SQ specification, the economic growth rate is reduced .26 percentage points by a fertility-induced population growth rate increase of one percentage point, whereas it is reduced .13 percentage points by a mortality-induced population growth rate change.

In Table 3 we assumed that households expect to work 2/3 of their expected lives, so when life expectancy increases, working life expectancy increases proportionately. However, in most developed countries, working life expectancy has not increased in proportion to life expectancy, with retirement ages remaining constant or even decreasing. In Table 4, we consider the same increases in life expectancy holding constant both the birth rate (at 3%) and the expected working life (at 40 years). Now increases in life expectancy imply longer retirement periods. When life expectancy is equal to 40 years (the same as working life expectancy), growth rates are near zero or negative in all specifications except the Blanchard specification.[41] Increased life expectancy (reduced mortality) increases saving and economic growth rates along with higher population growth rates in all specifications. In the Intermediate SQ specification, when life expectancy is increased from 40 years to 90 years (with a corresponding rise in retirement time), the economic growth rate increased by 2.79 percentage points, from -0.54 to 2.25%. The population growth rate is increased from 0.79% to 2.37% by the increase in life expectancy. This analysis suggests that the differences between the observed cross-section correlations between economic and population growth rates in less developed (higher fertility, negative correlation) and developed (lower mortality, positive correlation) countries may reflect the longer retirement periods associated with reduced mortality rather than some intrinsic difference between population growth rates fueled by higher fertility versus reduced mortality rates.

5.3 The Effect of Increased Longevity

Life expectancy continues to increase in most developed economies.[42] In Table 5, we isolate the effect of an increase in life expectancy on saving and economic growth rates by holding constant

[41] Computed growth and saving rates are negative in the Samuelson specification, indicating no viable equilibrium.
[42] In the United States, life expectancy at birth is currently increasing at about one and a half years per decade.

the population growth rate at our benchmark value of 1.5% and assuming that households expect to work 2/3 of their expected lives. Birth rates are reduced to satisfy the demographic constraint (25), and reported in the table. Initially, increases in life expectancy from a low value of 40 years increase saving and economic growth rates, but further increases above a critical value causes saving and economic growth rates to decrease. Interestingly, such an inverted U-shape relationship between economic growth rates and aging was found in a cross-country study by An and Jeong (2006).[43] In Table 5, the inverted U-shape relationship is most pronounced in the Samuelson specification, where an increase in life expectancy from 40 years to 60 years increases the economic growth rate from 1.81% to 2%, while a further increase to 90 years decreases the economic growth rate to 1.75%. Saving rates follow a similar inverted-U shape pattern. We find that this pattern is present in all specifications, although it is not apparent in the Blanchard specification in Table 5.

A plausible explanation for this observed inverted-U relationship between economic growth and longevity is as follows. With a short enough working period, increased longevity increases required saving rates because households choose rising consumption levels over their lives. The higher saving rate by young households increases saving and growth rates. However, if the working period increases proportionately with longevity, at some critical value the longer working period coupled with rising labor productivity over calendar time would allow younger households to finance retirement and higher future consumption levels with a lower saving rate, thereby decreasing the aggregate saving and economic growth rates.

5.4 The Effect of Lengthening the Retirement Time

As we saw in the discussion of Table 4, longer retirement times may explain positive correlations between economic and population growth rates found for developed economies. In Table 6 we isolate the effect of longer retirement by changing the ratio of the expected working life to life expectancy. We calibrate the model using our benchmark values of 1.5% for the population growth rate and 60 years for life expectancy. Using equation (26), we find the demographic

[43] From a historical perspective, Nicolini (2004) argued that the decline of adult mortality at the end of the 17th century was the cause of an increase in investment in pre-industrial England.

parameters required for the tabled expected working time ratio ψ. We also consider the case of no retirement by setting $L(x)=1$ at every age.

From Table 6, we see that longer retirement has a dramatic effect on saving and economic growth rates. With no retirement, economic growth rates are very low for all specifications. At the other extreme, where households expect to spend fully half of their lives retired, economic growth rates range from just under 2% in the Blanchard specification to over 4% in the Samuelson specification. Increased retirement time has the greatest impact on growth rates when the survivorship function is concave, as it is in developed countries. If the survivorship function is linear (Intermediate-L) or convex (Blanchard), longer retirement time increases the saving and growth rates, but to a lesser degree. The reason for this is that surviving to retirement is less certain with such survivorship functions. By comparison, under the Samuelson specification the household is certain of surviving to retirement. In the Intermediate SQ specification, the economic growth rate is .31% when households work all of their lives, but rises to 2.78% if households expect to work only half of their lives. Savings rates rise from 6% to 14.25%. The results in Table 6 suggest that the retirement savings motive may be an important growth driver in developed economies.[44]

5.5 A Malthusian Growth Trap

The above computations show that a country's demography can have a significant impact on its economic growth rate, particularly when the survivorship function is concave. Taken together, we find that the demographic effects can cause a "demographic" or "Malthusian" growth trap for economies where life expectancy is short, population growth rates are high due to high birth rates, and households spend little or no time in retirement. These are demographic characteristics found in many less developed economies.

In Table 7, we compute saving and economic growth rates for an economy where the population grows at 3%, life expectancy is 40 years (that is, 60 years at birth), and households do not retire. In all specifications, the economic growth rate is negative even though savings rates are

[44] This, in turn, suggests that pay-as-you-go Social Security may have important effects on growth to the extent that it reduces private retirement saving. We do not analyze such government policies in this paper.

positive range from about 4% to nearly 10%.[45] Again we find the most pronounced results in the Samuelson specification, where the aggregate saving rate is 3.78% and the economic growth rate is negative 1.87%. In the Blanchard specification, the saving rate is much higher, nearly 10%, and the economic growth rate is just negative. In all cases, the birth rates calculated from the demographic constraint (25) exceed 4%.

As mentioned earlier, a concave survivorship function is a characteristic of developed economies where mortality hazard rates remain low until households reach old age. In less developed economies, survivorship functions are likely to be less concave because of moderate mortality hazard rates for the non-elderly. For a less developed economy, the Intermediate-L, rather than the Intermediate-SQ, survivorship function may be the best approximation.[46] For this specification, we compute an economic growth rate of negative .56% and a saving rate of 8.15%.

6. Conclusions

We have developed a tractable, yet realistic, framework of equilibrium growth in which overlapping generations of households are born, work and save, retire and die. We have computed economic growth rates using standard production and taste parameters using a variety of demographic assumptions. While the model's demography is quite general, it includes the "classic" Blanchard and Samuelson OLG models, as well as the infinitely-lived representative agent growth model, as particular, limiting specifications of a two-parameter survivorship function.

Our analysis finds that demographic conditions, including the form (concavity) of the survivorship function, have significant effects on the saving and economic growth rates prevailing in an economy. As compared to the infinitely-lived agent growth model, the introduction of mortality *per se* reduces the aggregate saving and economic growth rates—more so when the survivorship function is more concave. On the other hand, mortality coupled with retirement increases the saving

[45] Saving rates are positive, and in some cases substantial, because a high population growth rate implies a larger proportion of young (saving) households in the population.

[46] There are numerous caveats in applying the demographic growth model of this paper to less developed economies. First, assuming technologies are the same, the AK model implies the same return on capital in developed and less developed economies. Due to capital scarcity, less developed economies may have higher rates of return on capital (although it is not clear that the higher return to capital in less developed economies is not simply a higher risk premium.) Second, the model presumes highly developed capital markets, including perfect life annuities markets. This too seems unlikely for less developed economies.

and growth rates. A higher population growth rate, whether caused by higher fertility or longer life expectancy (reduced mortality), decreases the economic growth rate but increases the saving rate. However, a higher population growth rate caused by longer life expectancy increases the economic growth rate when longer life expectancy leads to longer retirements. These observations can explain differences between developed and less developed economies in some cross-country correlations of population and economic growth rates. We also find an inverted-U relationship between life expectancy and economic growth rates, consistent with some cross-country evidence. Finally, we identify a potential Malthusian trap, in which economies with high fertility, short life expectancy, and no retirement may suffer low or negative economic growth rates despite having positive, and in some cases high, saving rates.

Figure 1: The De Moivre Survival Function

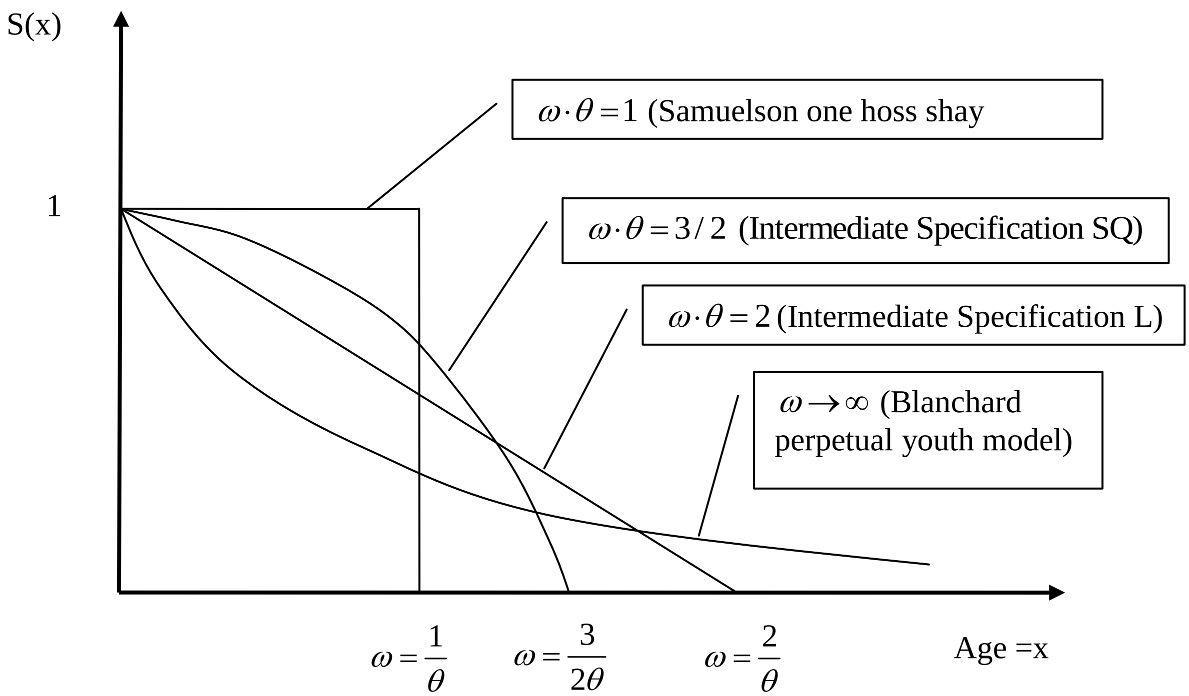

Figure 2: US Survival Functions, 1900 and 2003

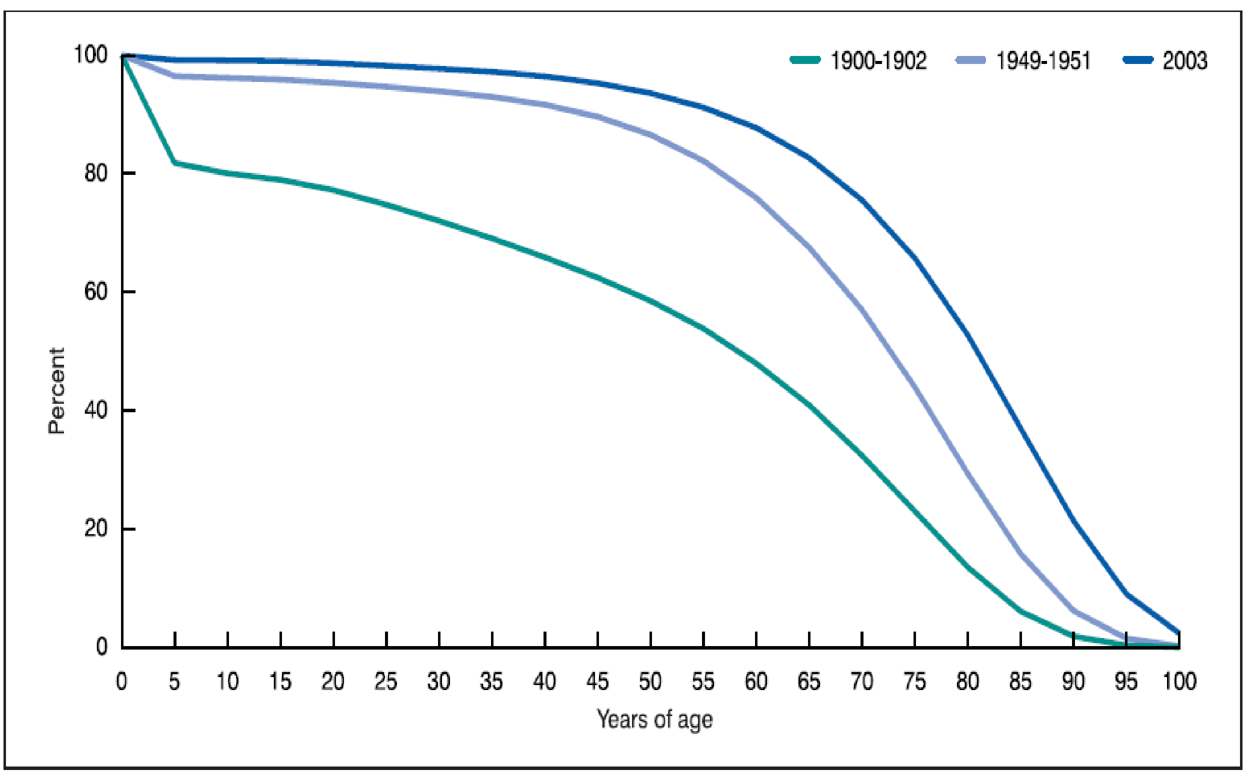

Figure 3. Percent surviving by age: Death-registration States, 1900–1902, and United States, 1949–51 and 2003

Source: National Vital Statistics Reports, Vol. 56 #9, December 28 2007.

Table 1: Effects of Mortality and Retirement on Saving and Economic Growth with Stationary Population

	Romer b=0	Blanchard b=1.67%	Intermediate L b=1.67%	Intermediate SQ b=1.67%	Samuelson b=1.67%
No Retirement $L(x)=1$	g=1.47% σ=4.89	g=1.07% σ=3.57	g=0.67% σ=2.22	g=0.48% σ=1.60	g~0 σ~0
$\psi=2/3$		g=1.69% σ=5.65	g=1.77% σ=5.89	g=1.88% σ=6.27	g=2.55% σ=8.51

Calibration: $A=0.35, \alpha=1/3, \rho=0.03, \delta=0.05, \varepsilon=-1.5$*; population growth (n)=0; in mortality cases life expectancy (χ) equals 60.*

Table 2: Effects of Population Growth from Increased Fertility on Saving and Economic Growth

population growth rate	Blanchard	Intermediate L	Intermediate SQ	Samuelson
n=0, b=1.67%	g=1.69% σ=5.65	g=1.77% σ=5.89	g=1.88% σ=6.27	g=2.55% σ=8.51
n=1.5%	g=1.29% σ=9.30 b=3.17	g=1.44% σ=9.49 b=2.80	g=1.51% σ=10.02 b=2.19	g=2.00% σ=11.66 b=2.53
n=3%	g=0.86% σ=12.85 b=4.67	g=1.06% σ=13.54 b=4.11	g=1.11% σ=13.69 b=3.93	g=1.45% σ=14.83 b=3.59

Calibration: $A=0.35, \alpha=1/3, \rho=0.03, \delta=0.05, \varepsilon=-1.5$*; life expectancy (χ) equals 60; expected working time ratio (*ψ*) equals to 2/3.*

Table 3: Effects of Population Growth from Increased Life Expectancy on Saving and Economic Growth

Life Expectancy	Blanchard	Intermediate L	Intermediate SQ	Samuelson
χ=40	g=1.55% σ=6.82 n=0.50	g=1.44% σ=7.2 n=0.72	g=1.47% σ=7.54 n=0.79	g=2.04% σ=9.95 n=0.94
χ=60	g=1.34% σ=8.9 n=1.33	g=1.38% σ=10.4 n=1.74	g=1.41% σ=10.99 n=.019	g=1.74% σ=13.14 n=.022
χ=90	g=1.19% σ=10.26 n=1.89	g=1.25% σ=11.78 n=2.28	g=1.26% σ=12.28 n=2.43	g=1.34% σ=13.64 n=2.75

Calibration: $A=0.35, \alpha=1/3, \rho=0.03, \delta=0.05, \varepsilon=-1.5$*; birth rate (b) equals 3%; expected working time ratio (*ψ*) equals to 2/3.*

Table 4: Effects of Life Expectancy on Saving and Economic Growth with Constant Birth Rate and Working Life

Life Expectancy	Blanchard	Intermediate L	Intermediate SQ	Samuelson
χ=40	g=2.40% σ=9.68 n=0.5	g=(.13)% σ=1.95 n=0.72	g=(0.54)% σ=0.84 n=0.79	No viable equilibrium
χ=60	g=2.52% σ=12.84 n=1.33	g=1.38% σ=10.40 n=1.74	g=1.41% σ=10.99 n=1.89	g=1.74% σ=13.14 n=2.20
χ=90	g=2.61% σ=14.98 n=1.89	g=2.04% σ=14.41 n=2.28	g=2.25% σ=15.39 n=2.37	g=2.92% σ=18.89 n=2.75

Calibration: $A = 0.35, \alpha = 1/3, \rho = 0.03, \delta = 0.05, \varepsilon = -1.5$*; birth rate (b) equals 3%, expected working life equals 40 years.*

Table 5: Effects of Life Expectancy on Saving and Economic Growth with Constant Population Growth Rate and Working Life Equal to 2/3.

Life Expectancy	Blanchard	Intermediate L	Intermediate SQ	Samuelson
χ=40	g=1.23% σ=9.11 b=4.0	g=1.24% σ=9.13 b=3.59	g=1.27% σ=9.22 b=3.49	g=1.81% σ=11.04 b=3.32
χ=60	g=1.29% σ=9.30 b=3.17	g=1.44% σ=9.79 b=2.80	g=1.51% σ=10.02 b=2.69	g=2.0% σ=11.66 b=2.53
χ=90	g=1.29% σ=9.29 b=2.61	g=1.43% σ=9.78 b=2.29	g=1.48% σ=9.93 b=2.19	g=1.75% σ=10.82 b=2.02

Calibration: $A = 0.35, \alpha = 1/3, \rho = 0.03, \delta = 0.05, \varepsilon = -1.5$*; population growth rate (n) equals 1.5%,expected working time ratio (*ψ*) equals to 2/3.*

Table 6: Effects of Working Time Ratio (ψ**) on Saving and Economic Growth**

Expected working time ratio	Blanchard b=3.17%	Intermediate L b=2.80%	Intermediate SQ b=2.69%	Samuelson b=2.53%
ψ =1/2	g=1.89% σ=11.29	g=2.40% σ=13.00	g=2.78% σ=14.25	g=4.02% σ=18.41
ψ =2/3	g=1.29% σ=9.30	g=1.44% σ=9.79	g=1.51% σ=10.02	g=2.00% σ=11.66
ψ =3/4	g=1.09% σ=8.63	g=1.07% σ=8.56	g=1.10% σ=8.68	g=1.29% σ=9.31
No Retirement $L(x)=1$	g=0.68% σ=7.28	g=0.46% σ=6.52	g=0.31% σ=6.00	g=(0.12%) σ=4.60

Calibration: $A = 0.35, \alpha = 1/3, \rho = 0.03, \delta = 0.05, \varepsilon = -1.5$*; population growth rate (n) equals 1.5%; life expectancy (χ) equals 60.*

Table 7: A Malthusian Growth Trap

	Blanchard	Intermediate L	Intermediate SQ	Samuelson
n=3% χ=40 No Retirement	g=(0.1%) σ=9.68 b=5.50	g=(0.56%) σ=8.15 b=4.83	g=(0.89%) σ=7.05 b=4.63	g=(1.87%) σ=3.78 b=4.29

Calibration: $A = 0.35, \alpha = 1/3, \rho = 0.03, \delta = 0.05, \varepsilon = -1.5$.

Appendix

A.1 Elimination of Solution $g = r - n$

Summing over surviving members of each cohort, aggregate consumption is

$$C_t = \int_{x=0}^{x=\omega} N(x)c(x)dx = \int_{x=0}^{x=\omega} N(x)\left\{w_t L(x) + i(x)f(x) - \frac{df(x)}{dx}\right\}dx$$

$$= w_t L_t + \int_{x=0}^{x=\omega} N(x)\left\{i(x)f(x) - \frac{df(x)}{dx}\right\}dx \tag{A.1}$$

From (21"), if $g = r - n$, then $C_t = w_t L_t$, so that (A.1) reduces to

$$\int_{x=0}^{x=\omega} N(x)\left\{i(x)f(x) - \frac{df(x)}{dx}\right\}dx = 0 \tag{A.2}$$

Substituting (12) yields

$$\int_{x=0}^{x=\omega} S(x)e^{-nx}\left\{i(x)f(x) - \frac{df(x)}{dx}\right\}dx = 0 \tag{A.3}$$

Integrating an individual agent's budget constraint (1b) over his lifetime, recognizing that his initial financial wealth is zero, and taking account of the transversality condition (3c), yields the inter-temporal constraint (8), which we may rewrite as

$$\int_{x=0}^{x=\omega} S(x)e^{-rx}\left\{w(x)L(x) - c(x)\right\}dx = 0 \tag{A.4}$$

so that the agent's present value of consumption equals the present value of his labor income. Substituting (1b) into (A.4) implies

$$\int_{x=0}^{x=\omega} S(x)e^{-rx}\left\{i(x)f(x) - \frac{df(x)}{dx}\right\}dx = 0 \tag{A.5}$$

Equations (A.3) and (A.5) can hold simultaneously if and only if $r = n$. But since for the present AK technology these are independently set parameters, there is no reason for this constraint to hold and hence $g = r - n$ is not a viable equilibrium.

A.2 Special Cases

In this Appendix, we present the equations for the two polar limiting cases of the general model and demonstrate that both models converge to the standard Romer growth model as lifetimes become infinite.

A.2.1 The Blanchard Perpetual Youth OLG Model

Taking the limit as $\omega, \ell \to \infty$ in equations (23a-(23e) and carrying out the integration yields

$$g+n=r+A\cdot(1-\alpha)\cdot\left[1-\frac{\Sigma^C}{\Sigma^L}\right] \tag{A.6a}$$

$$\Sigma^L=\frac{1}{n+\theta+\lambda} \tag{A.6b}$$

$$\Sigma^C=\frac{m(0)\cdot p(0)}{g+n-\left(\dfrac{r-\rho}{1-\varepsilon}\right)} \tag{A.6c)}$$

$$m(0)=\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}+\theta \tag{A.6d}$$

$$p(0)=\frac{1}{r+\theta+\lambda-g} \tag{A.6e}$$

$$r=\alpha\cdot A-\delta \tag{A.6f}$$

Substituting (A.6b)-(A.6e) into (A.6a), we obtain

$$g+n-r=A\cdot(1-\alpha)\cdot\left\{\frac{(g+n-r)\cdot\left[(\lambda+\theta+n)-\left(\dfrac{\rho-\varepsilon\cdot r}{1-\varepsilon}+\theta\right)-(g+n-r)\right]}{\left[(\lambda+\theta+n)-(g+n-r)\right]\cdot\left[(g+n-r)-\left(\dfrac{\rho-\varepsilon\cdot r}{1-\varepsilon}+\theta\right)\right]}\right\} \tag{A.6a'}$$

Because $g+n-r\neq 0$, so we can divide (A.6a') by $g+n-r$ to obtain a quadratic equation

$$\left[1-A\cdot(1-\alpha)\right]\cdot\left[(g+n-r)^2-\left(\lambda+n-\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\right)\cdot(g+n-r)\right]-(\lambda+\theta+n)\cdot\left(\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}+\theta\right)=0 \tag{A.7}$$

We can rewrite $g+n-r$ as $\hat{g}+\left(n-\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\right)$, where $\hat{g}=g-\frac{r-\rho}{1-\varepsilon}$ is the deviation of the growth rate from the representative agent rate, and rewrite (A.7) as

$$\hat{g}^2+\left(n-\lambda-\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\right)\cdot\hat{g}-\left[\lambda\cdot\left(n-\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\right)+\frac{\left(\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}+\theta\right)\cdot(\lambda+\theta+n)}{1-A\cdot(1-\alpha)}\right]=0. \tag{A.7'}$$

Solving (A.7'), we obtain

$$\hat{g}=\left[\frac{1}{2}\cdot\left(\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}-\lambda-n\right)\pm\sqrt{\frac{\left(\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}-\lambda-n\right)^2}{4}-\frac{(\lambda+\theta+n)\cdot\left(\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}+\theta\right)}{1-A\cdot(1-\alpha)}}\right] \tag{A.8}$$

From (A.8) we see that $\hat{g}\to 0$ as $\lambda+\theta+n\to 0$ (infinitely-lived representative agent model).

A.2.2 The Samuelson One-Hoss Shay Model

In this case, $\lambda\cdot\ell=1$ so $\left(1-\frac{x}{\ell}\right)^{\lambda\cdot\ell-1}=1$ for $x\in[0,\ell]$ and zero for $x>\ell$. Similarly, $\left(1-\frac{x}{\omega}\right)^{\theta\cdot\omega-1}=1$ for $x\in[0,\omega]$ and zero for $x>\theta$. Integrating, we obtain

$$\Sigma^L=\frac{1-e^{-n\cdot\ell}}{n} \tag{A.6b'}$$

$$\Sigma^C=m(0)\cdot p(0)\cdot\frac{1-e^{-\left(\frac{r-\rho}{1-\varepsilon}-g-n\right)\omega}}{g+n-\frac{r-\rho}{1-\varepsilon}} \tag{A.6c'}$$

$$m(0)=\frac{\rho-\varepsilon\cdot r}{(1-\varepsilon)\cdot\left(1-e^{-\left(\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\right)\cdot\omega}\right)} \tag{A.6d'}$$

$$p(0)=\frac{1-e^{-(r-g)\cdot\ell}}{r-g} \tag{A.6e'}$$

For simplicity, assume no retirement so $\ell=\omega$. Substituting into (A.6a) and rearranging we can write the solution as

$$\hat{g}+n=\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}+A\cdot(1-\alpha)\cdot\left[1-\frac{n\cdot\left(\frac{\rho-\varepsilon r}{1-\varepsilon}\right)\cdot\left(1-e^{-\left[\frac{\rho-\varepsilon r}{1-\varepsilon}-\hat{g}\right]\cdot\omega}\right)\cdot\left(1-e^{(\hat{g}+n)\omega}\right)}{\left(1-e^{-n\omega}\right)\cdot\left(1-e^{-\left(\frac{\rho-\varepsilon r}{1-\varepsilon}\right)\cdot\omega}\right)\left(\frac{\rho-\varepsilon r}{1-\varepsilon}-\hat{g}\right)\cdot(\hat{g}+n)}\right]. \qquad \text{(A.8')}$$

If we let $n\to 0$ (stable population) we can write[47]

$$\hat{g}^2=\left(\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}+A\cdot(1-\alpha)\right)\cdot\hat{g}+A\cdot(1-\alpha)\cdot\left[\frac{\left(\frac{\rho-\varepsilon r}{1-\varepsilon}\right)\cdot\left(1-e^{-\left[\frac{\rho-\varepsilon r}{1-\varepsilon}-\hat{g}\right]\cdot\omega}\right)\cdot\left(1-e^{\hat{g}\omega}\right)}{\omega\cdot\left(1-e^{-\left(\frac{\rho-\varepsilon r}{1-\varepsilon}\right)\cdot\omega}\right)\left(\frac{\rho-\varepsilon r}{1-\varepsilon}-\hat{g}\right)}\right]. \qquad \text{(A.8'')}$$

As we let $\omega\to\infty$, the last term on the RHS of (A.8'') vanishes, and the root of the remaining equation is $\hat{g}=0$. That is, the Samuelson model growth rate converges to that of the representative agent model as we increase the lifetime of the agent.

[47] We use $\lim_{n\to 0}\frac{n}{1-e^{-n\omega}}=\frac{1}{\omega}$

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